

Forward transformation

The transformation can be defined by the sequential multiplication of the matrices:

$${}^w A_t = {}^w A_O \cdot {}^O A_A \cdot {}^A A_P \cdot {}^P A_t \quad (18)$$

with the matrices built up as follows:

$${}^w A_O = \begin{bmatrix} C_B & 0 & -S_B & 0 \\ 0 & 1 & 0 & 0 \\ S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^O A_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & D_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (19)$$

$${}^A A_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_A & S_A & 0 \\ 0 & -S_A & C_A & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^P A_t = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y - D_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

Executing the matrix multiplications starting from the left:

$${}^w A_A = {}^w A_O \cdot {}^O A_A$$

$${}^w A_A = \begin{bmatrix} C_B & 0 & -S_B & 0 \\ 0 & 1 & 0 & 0 \\ S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & D_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_B & 0 & -S_B & 0 \\ 0 & 1 & 0 & D_y \\ S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^w A_P = {}^w A_A \cdot {}^A A_P$$

$${}^w A_P = \begin{bmatrix} C_B & 0 & -S_B & 0 \\ 0 & 1 & 0 & D_y \\ S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_A & S_A & 0 \\ 0 & -S_A & C_A & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_B & S_A S_B & -S_B C_A & 0 \\ 0 & C_A & S_A & D_y \\ S_B & -S_A C_B & C_A C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^w A_t = {}^w A_P \cdot {}^P A_t$$

$${}^w A_t = \begin{bmatrix} C_B & S_A S_B & -S_B C_A & 0 \\ 0 & C_A & S_A & D_y \\ S_B & -S_A C_B & C_A C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y - D_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^w A_t = \begin{bmatrix} C_B & S_A S_B & -S_B C_A & C_B P_x + S_A S_B (P_y - D_y) - S_B C_A P_z \\ 0 & C_A & S_A & C_A (P_y - D_y) + S_A P_z + D_y \\ S_B & -S_A C_B & C_A C_B & S_B P_x - S_A C_B (P_y - D_y) + C_A C_B P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (21)$$

$$Q = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_B P_x + S_A S_B (P_y - D_y) - S_B C_A P_z \\ C_A (P_y - D_y) + S_A P_z + D_y \\ S_B P_x - S_A C_B (P_y - D_y) + C_A C_B P_z \\ 1 \end{bmatrix} \quad (25)$$

$$Q = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_B & S_A S_B & -S_B C_A & -S_A S_B D_y \\ 0 & C_A & S_A & -C_A D_y + D_y \\ S_B & -S_A C_B & C_A C_B & S_A C_B D_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} = {}^Q A_P \cdot P \quad (26)$$

Inverse transformation

Using ${}^q A_p = \begin{bmatrix} R & q \\ 0 & 1 \end{bmatrix}$

and (26)

We extract the rotation matrix R and the translation vector q :

$$R = \begin{bmatrix} C_B & S_A S_B & -S_B C_A \\ 0 & C_A & S_A \\ S_B & -S_A C_B & C_A C_B \end{bmatrix} \quad q = \begin{bmatrix} -S_A S_B D_y \\ -C_A D_y + D_y \\ S_A C_B D_y \end{bmatrix}$$

Using $({}^q A_p)^{-1} = \begin{bmatrix} R^T & -R^T q \\ 0 & 1 \end{bmatrix}$

$$-R^T = \begin{bmatrix} -C_B & 0 & -S_B \\ -S_A S_B & -C_A & S_A C_B \\ S_B C_A & -S_A & -C_A C_B \end{bmatrix}$$

$$-R^T q = \begin{bmatrix} -C_B & 0 & -S_B \\ -S_A S_B & -C_A & S_A C_B \\ S_B C_A & -S_A & -C_A C_B \end{bmatrix} \cdot \begin{bmatrix} -S_A S_B D_y \\ -C_A D_y - D_y \\ S_A C_B D_y \end{bmatrix} = \begin{bmatrix} 0 \\ S_A^2 S_B^2 D_y + C_A^2 D_y + C_A D_y + S_A^2 C_B^2 D_y \\ -S_A S_B^2 C_A D_y + S_A C_A D_y + S_A D_y - S_A C_A C_B^2 D_y \end{bmatrix}$$

$$P = ({}^Q A_p)^{-1} \cdot Q$$

$$P = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_B & 0 & S_B & 0 \\ S_A S_B & C_A & -S_A C_B & S_A^2 S_B^2 D_y + C_A^2 D_y + C_A D_y + S_A^2 C_B^2 D_y \\ -S_B C_A & S_A & C_A C_B & -S_A S_B^2 C_A D_y + S_A C_A D_y + S_A D_y - S_A C_A C_B^2 D_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{bmatrix} \quad (27)$$

$$P_x = C_B Q_x + S_B Q_z$$

$$P_y = S_A S_B Q_x + C_A Q_y - S_A C_B Q_z + S_A^2 S_B^2 D_y + C_A^2 D_y + C_A D_y + S_A^2 C_B^2 D_y$$

$$P_z = -S_B C_B Q_x + S_A Q_y + C_A C_B Q_z - S_A S_B^2 C_A D_y + S_A C_A D_y + S_A D_y - S_A C_A C_B^2 D_y$$

(28)